Incompressible flow with heat transfer

We recall heat transfer for a solid:

\[ \frac{\partial \theta}{\partial t} + \mathbf{q}^B = 0 \quad \text{in } V \]  

(16.1)

\[ \theta \big|_{S_\theta} \text{ is prescribed, } k \frac{\partial \theta}{\partial n} \big|_{S_q} = q^S \big|_{S_q} \]  

(16.2)

\[ S_\theta \cup S_q = S \quad S_\theta \cap S_q = \emptyset \]  

(16.3)

**Principle of virtual temperatures**

\[ \int_V \bar{\theta}_i k \theta_i dV = \int_V \bar{\theta}q^B dV + \int_{S_q} \bar{\theta}^S q^S dS_q \]  

(16.4)

for arbitrary continuous \( \bar{\theta}(x_1, x_2, x_3) \) zero on \( S_\theta \)

For a fluid, we use the Eulerian formulation.
\[ \rho c_p \theta \frac{\partial}{\partial x} + \frac{\partial}{\partial x} (\rho c_p \theta) dx \]  + conduction + etc  

(16.5)

In general 3D, we have an additional term for the left hand side of (16.1):

\[ - \nabla \cdot (\rho c_p \theta v) = - \rho c_p \nabla \cdot (v \theta) = - \rho c_p (\nabla \cdot \mathbf{v}) \theta - \rho c_p (v \cdot \nabla) \theta \]

(16.6)

where \( \nabla \cdot \mathbf{v} = 0 \) in the incompressible case.

\[ \nabla \cdot \mathbf{v} = v_{i,i} = \text{div}(\mathbf{v}) = 0 \]

(16.7)

So (16.1) becomes

\[ (k \theta)_i + q^B = \rho c_p \theta_i v_i \Rightarrow (k \theta)_i + (q^B - \rho c_p \theta_i v_i) = 0 \]

(16.8)

Principle of virtual temperatures is now (use (16.4))

\[ \int_V \mathbf{T}_i k \theta_i dV + \int_V \mathbf{T}_i (\rho c_p \theta_i v_i) dV = \int_V \mathbf{T}_i q^B dV + \int_{S_q} \mathbf{T}_i q^S dS_q \]

(16.9)

Navier-Stokes equations

- **Differential form**
  \[ \tau_{ij,j} + f_{i}^B = \rho v_{i,j} v_j \]
  (16.10)
  
  with \( \rho v_{i,j} v_j \) like term (A) in (16.6) = \( \rho (\mathbf{v} \cdot \nabla) \mathbf{v} \) in \( V \).
  \[ \tau_{ij} = -p \delta_{ij} + 2\mu e_{ij} \quad e_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \]
  (16.11)

- **Boundary conditions** (need be modified for various flow conditions)
  \[ \tau_{ij} n_j = f_i^{S_f} \text{ on } S_f \]
  (16.12)

Mostly used as \( f_n = \tau_{nn} \) = prescribed, \( f_i = \) unknown with possibly \( \frac{\partial v_i}{\partial n} = \frac{\partial v_i}{\partial n} = 0 \) (outflow or inflow conditions).

\[ \mathbf{T} \]

\[ \mathbf{f}/\mathbf{n} \text{ d boundary} \]

And \( v_i \) prescribed on \( S_v \), and \( S_v \cup S_f = S \) and \( S_v \cap S_f = \emptyset \).
• Variational form

\[ \int_V \rho v_i \rho v_i, j v_j dV + \int_V \tau_{ij} \tau_{ij} dV = \int_V \mathbf{v}_i f_i^B dV + \int_{S_f} \mathbf{v}_i S_f^S dS_f \]  

(16.13)

\[ \int_V p \nabla \cdot \mathbf{v} dV = 0 \]  

(16.14)

• F.E. solution

We interpolate \((x_1, x_2, x_3), v_i, \mathbf{v}_i, \theta, \bar{\theta}, p, \bar{p}\). Good elements are

\[ \times: \text{linear pressure} \]

\[ \circ: \text{biquadratic velocities} \]

\((Q_2, P_1), 9/3\) element

\[ 9/4c\) element

Both satisfy the inf-sup condition.

So in general,

Example:

For \(S_f\) e.g.

\[ \tau_{nn} = 0, \quad \frac{\partial v_i}{\partial n} = 0; \]  

(16.15)
and $\frac{\partial v}{\partial t}$ is solved for. Actually, we frequently just set $p = 0$.

Frequently used is the 4-node element with constant pressure

It does not strictly satisfy the inf-sup condition. Or use 3-node element with a bubble node. Satisfies inf-sup condition

1D case of heat transfer with fluid flow, $v = \text{constant}$

\[
\text{Re} = \frac{vL}{\nu} \quad \text{Pe} = \frac{vL}{\alpha} \quad \alpha = \frac{k}{\rho c_p}
\]  \hspace{1cm} (16.16)

- **Differential equations**

\[
k\theta'' = \rho c_p \theta' v
\]  \hspace{1cm} (16.17)

\[
\theta|_{x=0} = \theta_L \quad \theta|_{x=L} = \theta_R
\]  \hspace{1cm} (16.18)

In non-dimensional form

\[
\frac{1}{\text{Pe}}\theta'' = \theta'
\]  \hspace{1cm} (now $\theta''$ and $\theta'$ are non-dimensional) \hspace{1cm} (16.19)

\[
\Rightarrow \frac{\theta - \theta_L}{\theta_R - \theta_L} = \frac{\exp \left(\frac{\text{Pe}}{T}x\right) - 1}{\exp(\text{Pe}) - 1}
\]  \hspace{1cm} (16.20)
\[ \theta'' = \text{Pe} \theta' \]  

(16.21)

\[ \int_0^1 \theta' \theta' \, dx + \text{Pe} \int_0^1 \theta \theta' \, dx = 0 + \{ \text{effect of boundary conditions} = 0 \text{ here} \} \]  

(16.22)

Using 2-node elements gives

\[ \frac{1}{(h^*)^2} (\theta_{i+1} - 2\theta_i + \theta_{i-1}) = \frac{\text{Pe}}{2h^*} (\theta_{i+1} - \theta_{i-1}) \]  

(16.23)

\[ \text{Pe} = \frac{vL}{\alpha} \]  

(16.24)

Define

\[ \text{Pe}^c = \text{Pe} \cdot \frac{h}{L} = \frac{vh}{\alpha} \]  

(16.25)

\[ \left(-1 - \frac{\text{Pe}^c}{2}\right) \theta_{i-1} + 2\theta_i + \left(\frac{\text{Pe}^c}{2} - 1\right) \theta_{i+1} = 0 \]  

(16.26)

what is happening when \( \text{Pe}^c \) is large? Assume two 2-node elements only.

\[ \theta_{i-1} = 0 \]  

(16.27)

\[ \theta_{i+1} = 1 \]  

(16.28)

\[ \theta_i = \frac{1}{2} \left(1 - \frac{\text{Pe}^c}{2}\right) \]  

(16.29)
For $Pe^e > 2$, we have negative $\theta_i$ (unreasonable).