Timoshenko beam theory

The fiber moves up and rotates and its length does not change.

**Principle of virtual displacement** (Linear Analysis)

\[
EI \int_0^L \left( \frac{\beta'}{\beta} \right)^T \beta' \, dx + (Ak)G \int_0^L \left( \frac{dw}{dx} - \beta \right)^T \left( \frac{dw}{dx} - \beta \right) \, dx = \int_0^L \bar{w}^T p \, dx
\]  

(20.1)

Two-node element:

Three-node element:

For a \( q \)-node element,

\[
\hat{u} = \begin{bmatrix} w_1 & \cdots & w_q & \theta_1 & \cdots & \theta_q \end{bmatrix}^T
\]  

(20.2)

\[
w = H_w \hat{u}
\]  

(20.3)

\[
\beta = H_\beta \hat{u}
\]  

(20.4)

\[
H_w = \begin{bmatrix} h_1 & \cdots & h_q & 0 & \cdots & 0 \end{bmatrix}
\]  

(20.5)

\[
H_\beta = \begin{bmatrix} 0 & \cdots & 0 & h_1 & \cdots & h_q \end{bmatrix}
\]  

(20.6)

\[
J = \frac{dx}{dr}
\]  

(20.7)
\[
\frac{dw}{dx} = J^{-1} H_{w,r} \hat{u} \tag{20.8}
\]
\[
\frac{d\beta}{dx} = J^{-1} H_{\beta,r} \hat{u} \tag{20.9}
\]

Hence we obtain

\[
\left[ EI \int_{-1}^{1} B_{\beta}^T B_{\beta} \det(J) \, dr + (Ak)G \int_{-1}^{1} (B_{w} - H_{\beta})^T (B_{w} - H_{\beta}) \det(J) \, dr \right] \hat{u} = \int_{-1}^{1} H_{w,p}^T \det(J) \, dr \tag{20.10}
\]

\[
K \hat{u} = R \tag{20.11}
\]

\( K \) is a result of the term inside the bracket in (20.10) and \( R \) is a result of the right hand side.

For the 2-node element,

\[
w_1 = \theta_1 = 0 \tag{20.12}
\]
\[
w_2, \theta_2 = ? \tag{20.13}
\]
\[
\gamma = \frac{w_2}{L} - \frac{1 + r}{2} \theta_2 \tag{20.14}
\]

We cannot make \( \gamma \) equal to zero for every \( r \) (page 404, textbook). Because of this, we need to use about 200 elements to get an error of 10%. (Not good!)

Recall almost or fully incompressible analysis: Principle of virtual displacements:

\[
\int_V \varepsilon^T C' \varepsilon' \, dV + \int_V \tau_v (\kappa \epsilon_v) \, dV = R \tag{20.15}
\]

\( u/p \) formulation

\[
\int_V \varepsilon^T C' \varepsilon' \, dV - \int_V \tau_v \, p \, dV = R \tag{20.16}
\]
\[
\int_V p \left( \frac{p}{\kappa} + \epsilon_v \right) \, dV = 0 \tag{20.17}
\]

But now we needed to select wisely the interpolations of \( u \) and \( p \). We needed to satisfy the inf-sup condition

\[
\inf_{\text{u}_h \in V_h} \sup_{\text{q}_h \in Q_h} \frac{\int_{V_h} \text{u}_h \nabla \cdot \text{v}_h \, dVol}{\|\text{q}_h\| \|\text{v}_h\|} \geq \beta > 0 \tag{20.18}
\]
4/1 element:

We can show mathematically that this element does not satisfy inf-sup condition. But, we can also show it by giving an example of this element which violates the inf-sup condition.

\[ v_1 = \Delta, \quad v_2 = 0 \Rightarrow \nabla \cdot v_h \text{ for both elements is positive and the same. Now, if I choose pressures as above} \]

\[ \int_{V_{ol}} q_h \nabla v_h dVol = 0, \quad \text{hence (20.18) is not satisfied!} \]  

9/3 element

satisfies inf-sup

9/4-c

satisfies inf-sup

Getting back to beams

\[ EI \int_0^L \bar{\beta} \beta dx + (AkG) \int_0^L \left( \frac{d\bar{\gamma}}{dx} - \bar{\beta} \right) \gamma^{AS} dx = R \]

(20.20)

\[ \int_0^L \gamma^{AS} (\gamma - \gamma^{AS}) dx = 0 \]

(20.21)

where

\[ \gamma = \frac{dw}{dx} - \beta, \quad \text{from displacement interpolation} \]

(20.22)
\[ \gamma^{AS} = \text{Assumed shear strain interpolation} \]  
\[ (20.23) \]

2-node element, constant shear assumption. From (20.21),

\[ \int_0^L \left( \frac{dw}{dx} - \beta \right) \gamma^{AS} dx = \int_0^L \gamma^{AS} \gamma^{AS} dx \]
\[ (20.24) \]

\[ \Rightarrow - \int_{-1}^{+1} \left( \frac{1 + r}{2} \right) \cdot \frac{L}{2} dr + w_2 = \gamma^{AS} \cdot L \]
\[ (20.25) \]

\[ \Rightarrow \gamma^{AS} = \frac{w_2 - L \theta_2}{L} \]
\[ (20.26) \]

\( \gamma^{AS} \) (shear strain) is equal to the displacement-based shear strain at the middle of the beam.

Use \( \gamma^{AS} \) in (20.20) to obtain a powerful element. For “our problem”,

\[ \gamma^{AS} = 0 \quad \text{hence} \quad w_2 = \frac{L}{2} \theta_2 \]
\[ (20.27) \]

\[ \Rightarrow EI \int_0^L \beta \beta' dx = M \left. \beta \right|_{x=L} \]
\[ (20.28) \]

\[ \Rightarrow EI \left( \frac{1}{L} \right)^2 \cdot L \theta_2 = M \]
\[ (20.29) \]

\[ \Rightarrow \theta_2 = \frac{ML}{EI}, \quad w_2 = \frac{ML^2}{2EI} \]
\[ (20.30) \]

(exact solutions)
Plates

\[
\begin{align*}
  w &= w(x, y) \quad \text{is the transverse displacement of the mid-surface} \\
  v &= -z\beta_y(x, y) \\
  u &= -z\beta_x(x, y)
\end{align*}
\]  

(20.31)

For any particle in the plate with coordinates \((x, y, z)\), the expressions in (20.31) hold!

We use

\[
\begin{align*}
  w &= \sum_{i=1}^{q} h_i w_i \\
  \beta_x &= -\sum_{i=1}^{q} h_i \theta_y^i \\
  \beta_y &= +\sum_{i=1}^{q} h_i \theta_x^i
\end{align*}
\]

(20.32) \hspace{1cm} (20.33) \hspace{1cm} (20.34)

where \(q\) equals the number of nodes. Then the element locks in the same way as the displacement-based beam element.